are assumed to depend only on $(t, \mathbf{x})$.

If the function μ_m is upper (lower) semicontinuous, then by /4/ only the second (first) player in general has an optimal strategy. Therefore, in this case, we only have a pursuit (evasion) problem. We see that Proposition 1 on pursuit (Proposition 2 on evasion) of a fuzzy set starting from a fuzzy starting position holds in this case also.

I would like to thank A.I. Subbotin for useful comments.

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ON THE SMALL VIBRATIONS OF A STRATIFIED CAPILLARY LIQUID*

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An initial-boundary value problem is considered for the equation which describes the vibrations of an ideal, stratified liquid occupying the lower half space in the Boussinesq approximation. The Vaisala-Brunt frequency is assumed to be constant. The boundary condition on the planar boundary is a combination of the conditions on the solid cover and the free surface and, moreover, the latter contains a term which takes account of capillarity. A formulation of the problem is given, its solution is constructed and its behaviour at long times is investigated.

1. Formulation of the problem. Let us assume that the stratified liquid occupies the half space $R\_^3 = \{x = (x_1, x_2, x_3) \mid x_3 < 0\}$. We denote by Π_1 the part of the plane $x_3 = 0$ which is the surface of contact between the liquid and the solid cover and, by Π_0, the set of points with which the free surface of the liquid in the unperturbed state coincides, that is, $\Pi_0 = \{x | x_3 = 0\} \setminus \Pi_1$. For convenience, we introduce into R^2 the sets Σ_0 and Σ_1 which are associated with Π_0 and Π_1 in the following manner:

$$\Pi_i = \{x \mid x_3 = 0 \wedge x' = (x_1, x_2) \in \Sigma_i\} \quad (i = 0, 1)$$

We require that Σ_0 should be a bounded domain with a smooth boundary $\partial\Sigma_0$. Let us consider the following problem:

$$\Delta_3 u_{tt} + \omega_0^2 \Delta_2 u = 0, \quad x_3 < 0, \quad t > 0 \tag{1.1}$$

$$u(x, 0) = u_t(x, 0) = 0 \tag{1.2}$$

*Prikl.Matem.Mekhan.,53,1,66-73,1989

$$u_{x_3}|_{\Pi_1} = 0 \qquad (1.3)$$

$$(1 - \sigma\Delta_2) u_{x_3} + (\partial^2/\partial t^2 + \omega_0^2) u|_{\Pi_0} = f(x', t)$$

$$\partial u_{x_3}/\partial n + \varkappa u_{x_3}|_{\partial\Pi_0} = 0 \qquad (1.4)$$

Here

$$f(x', t) \in C^{(3)}([0, T], W_2^{-1}(\Sigma_0)) \cap C_0^{(2)}([0, T], W_2^{-1}(\Sigma_0))$$

$$\forall T > 0; \quad \Delta_l = \sum_{k=1}^{l} \frac{\partial^2}{\partial x_k^2}$$

$$\partial\Pi_0 = \{x \mid x_3 = 0 \wedge x' \in \partial\Sigma_0\}$$

$$C_0^{(k)}([0, T], H) = \{\varphi(x', t) \mid \varphi \in C^{(k)}([0, T], H) \wedge D_t^l \varphi(x', t = 0) = 0 \ (l = 0, \ldots, k-1)\}$$

The meaning of the scalar function $u(x, t)$ has been described in /1/. The boundary condition (1.3) is the condition that the vertical component of the velocity field on the solid cover should vanish. The first condition of (1.4) corresponds to the specification of a perturbation on the free surface and the term $\sigma\Delta_2 u_{x_3}$ corresponds to the capillarity. The second equality of (1.4) is the condition for the conservation of the wetting angle during the course of the motion /2/ (it is assumed that the solid cover has a finite thickness and that the liquid does not overflow across its upper edge).

Hence, the small motions of a stratified, capillary liquid occupying a basin are modelled. The basin may be considered as being unbounded and infinitely deep below an ice field Π_1 having a part of its surface Π_0 free from ice.

The limits of applicability of the model being considered are determined by the usual requirements of a linearized model and the Boussinesq approximation. The condition on the free surface in dimensional variable has the form

$$(g - \sigma\rho_0^{-1}(0)\Delta_2) u_{x_3} + (\partial^2/\partial t^2 + \omega_0^2) u|_{\Pi_0} = f(x', t)$$

The case when $\sigma = 0$ (gravitational surfaces and internal waves) has been considered in /3/. Next, it is everywhere assumed that $\sigma > 0$, that is, gravitational-capillary waves are considered.

We shall call the set of all functions $v(x, t)$, defined on $R_-^3 \times [0, \infty)$ such that

$$\frac{\partial^{k+l+m} v(x, t)}{\partial t^k \partial x_i^l \partial x_j^m} \in C^{(0)}(Q \times [0, T]),$$

$$k = 0, 1, 2; \quad l, m = 0, 1; \quad i, j = 1, 2, 3$$

for any compact $Q \in R_-^3$ and any $T > 0$, the class of smoothness K.

We shall assume that the function $u(x, t)$, which is regular at infinity /1/, belongs to the class of smoothness K and satisfies Eq.(1.1) and the boundary conditions (1.2) in the classical sense and the boundary conditions (1.3) and (1.4) in a sense defined below, is the solution of the problem in question.

Definition. A function $u(x, t)$ from the smoothness class satisfies the boundary conditions (1.3) and (1.4) if, simultaneously,

1) a function $d_{x_3}u(x', t)$ exists which possesses the properties: $d_{x_3}u(x', t) \in W_2^1(\Sigma_0)$, $\forall t \geqslant 0$, $d_{x_3}u(x', t) = 0$ when $x' \in \Sigma_1$, $t \geqslant 0$ uniformly with respect to $t \in [0, T]$, $\forall T > 0$; $\| d_{x_3}u(x', t) - u_{x_3}(x, t)\|_{L_2(S)} \to 0$ as $x_3 \to 0$ (S is an arbitrary compact in R^2);

2) a function $\gamma u(x', t) \in C_0^{(2)}([0, T], L_2(S))$ exists such that

$$\| \partial^k u(x, t)/\partial t^k - D_t^k \gamma u(x', t)\|_{L_2(S)} \to 0$$

uniformly with respect to $t \in [0, T]$ for any $T > 0$ as $x_3 \to 0$ (S is an arbitrary compact in R^2; $k = 0, 1, 2$);

3) the equality

$$[d_{x_3}u, \eta] + ((D_t^2 + \omega_0^2)\gamma u, \eta)_{L_2(\Sigma_0)} = (f, \eta) \qquad (1.5)$$

$$\forall \eta \in W_2^1(\Sigma_0)$$

holds.

In (1.5) (f, η) is the result of the action of the functional $f \in W_2^{-1}(\Sigma_0)$ on the function. A scalar product in $W_2^1(\Sigma_0)$ is denoted by $[\ ,\ ]$

$$[\varphi, \eta] = (\varphi, \eta)_{L_2(\Sigma_0)} + \sigma (\nabla\varphi, \nabla\eta)_{L_2(\Sigma_0)} + \sigma\varkappa (\varphi, \eta)_{L_2(\partial\Sigma_0)}$$

We note that, if

$$d_{x_3}u (x', t) \in C^{(2)} (\Sigma_0), \quad f (x', t) \in L_2 (\Sigma_0), \quad \forall t \geqslant 0$$

satisfaction of Eq. (1.5) implies the conditions

$$(1 - \sigma\Delta_2)\, d_{x_3}u + (D_t^2 + \omega_0^2)\, \gamma u\, |_{\Sigma_0} = f (x', t)$$
$$\partial (d_{x_3}u)/\partial n + \varkappa d_{x_3}u\, |_{\partial\Sigma_0} = 0$$

Hence, it is assumed that the boundary conditions are satisfied in a weak sense.
Since the inequalities

$$|\, (f, \varphi)_{L_2(\Sigma_0)}\, | \leqslant \| f \|_{L_2(\Sigma_0)} \| \varphi \|_{L(\Sigma_0)} \leqslant C \| \varphi \|_+$$

and $(\| \varphi \|_+ = [\varphi, \varphi]^{1/2})$ hold when $\varphi \in W_2^1 (\Sigma_0)$ and $f \in L_2 (\Sigma_0)$, it follows from Riesz's theorem that an element $Gf \in W_2^1 (\Sigma_0)$ exists which is unique such that $(f, \varphi)_{L_2(\Sigma_0)} = [Gf, \varphi]$ and, moreover,

$$\| Gf \|_+ = \| f \|_- = \sup_{\varphi \in W_2^1(\Sigma_0)} \frac{|\, (f, \varphi)_{L_2(\Sigma_0)}\, |}{\| \varphi \|_+}$$

By virtue of this equality, a continuous extension of G on the whole of $W_2^{-1} (\Sigma_0)$ exists for which we retain the earlier notation. It is possible to set up the chain of inequalities

$$\| Gf \|_{L_2(\Sigma_0)} \leqslant \| Gf \|_+ \leqslant \| f \|_{L_2(\Sigma_0)} \leqslant \| f \|_+, \quad f \in W_2^1 (\Sigma_0) \tag{1.6}$$

It follows from (1.6) that

$$G (D_t^k \gamma u (x', t)) = D_t^k G\gamma u (x', t) \; (k = 0, 1, 2)$$
$$Gf (x', t) \in C^{(k)} ([0, T], \; W_2^1 (\Sigma_0))$$

when $f (x', t) \in C^{(k)} ([0, T], \; W_2^{-1} (\Sigma_0))$.
It can be shown that $\ker G = 0$.
By making use of the operator G, we obtain from (1.5) the equivalent condition

$$[d_{x_3}u + (D_t^2 + \omega_0^2)\, G\gamma u, \eta] = [Gf, \eta], \quad \forall \eta \in W_2^1 (\Sigma_0)$$

and the equivalent equation in $W_2^1 (\Sigma_0)$

$$d_{x_3}u + (D_t^2 + \omega_0^2)\, G\gamma u = Gf \tag{1.7}$$

The following theorem holds subject to the condition for a solution to exist.

Theorem 1. The solution of the problem in question is unique.
The proof is carried out using the energy relationship*. (*This relationship can be obtained as in the paper: Simakov S.T., On the theory of internal and surface waves, Moscow, 1987, 45 pp.; deposited in The All-Union Institute for Scientific and Technical Information (VINITI), 8814-B87, 16.12.87.).

2. Construction of the solution.

We shall seek the solution in the form

$$u (x, t) = P [\nu] (x, t) = P_1 [\nu] (x, t) + P_2 [\nu] (x, t) \tag{2.1}$$
$$P_1 [\nu] (x, t) = \frac{1}{2\pi} \int_{\Sigma_0} \frac{\nu (y', t)}{| x - y' |}\, dy'$$
$$P_2 [\nu] (x, t) = -\frac{\omega_0 | x_3 |}{2\pi} \int_{\Sigma_0} \int_0^t J_1 \left(\omega_0 \frac{| x_3 | (t - \tau)}{| x - y' |}\right) \frac{\nu (y', \tau)}{| x - y' |^2}\, d\tau\, dy',$$
$$\nu (y', t) \in C_0^{(2)} ([0, T], W_2^1 (\Sigma_0)), \; \forall T > 0$$
$$| x - y' | = (| x' - y' |^2 + x_3^2)^{1/2}, \quad | x' - y' | = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{1/2}$$

($J_1 (z)$ is a first-order Bessel function).

It can be verified that the function $u (x, t)$ defined in this manner is regular at infinity, belongs to the smoothness class K and satisfies Eq. (1.1) and conditions (1.2) in the classical sense and the conditions 1) and 2) cited in the definition. Moreover, the formulae

$$d_{x_3}u (x', t) = \nu (x', t) - \omega_0 S_{\omega_0 t} * \nu (x', t)$$

$$D_t^k\,(\gamma u\,(x',t)) = D_t^k V\,[\nu]\,(x',t) \quad (k=0,1,2)$$
$$S_{\omega_0 t} = -\int_0^{\omega_0 t} J_1(\alpha)\,\alpha^{-1}\,d\alpha$$

hold in $W_2^1(\Sigma_0)$

Here V is the continuous extension of the integral $(2\pi)^{-1}\int_{\Sigma_0} \nu(y')\,|\,x'-y'\,|^{-1}\,dy'$ on $L_2(\Sigma_0)$ which is considered as an operator on $C_0^\infty(\Sigma_0)$ in $L_2(\Sigma_0)$. An asterisk denotes convolution with respect to time.

From (1.7), we obtain the equation for $\nu(x',t)$

$$\nu - \omega_0 S_{\omega_0 t} * \nu + (D_t^2 + \omega_0^2)\,GV\nu = Gf \qquad (2.2)$$
$$\nu(x',t) \in C_0^{(2)}([0,T],\ W_2^1(\Sigma_0)),\quad \forall T > 0$$

It is known /5/ that V is a bound operator from $L_2(\Sigma_0)$ in $W_2^1(\Sigma_0)$. It can be shown that $\ker V = 0$ and

$$(V\varphi, \varphi)_{L_2(\Sigma_0)} \geqslant 0,\quad \forall\varphi \in L_2(\Sigma_0)$$

It is obvious from the definition of the operator G that it is symmetric and positive in $W_2^1(\Sigma_0)$. By virtue of this and the inequalities (1.6), we conclude /4/ that there exists a square root $G^{1/2}$ of the operator G which is bounded, positive and symmetric operator in $W_2^1(\Sigma_0)$. The equality

$$\|\,G^{1/2}\varphi\,\|_+ = \|\,\varphi\,\|_{L_2(\Sigma_0)},\quad \forall\varphi \in W_2^1(\Sigma_0)$$

is obvious.

As before, the extension of $G^{1/2}$ with respect to continuity onto the whole of $L_2(\Sigma_0)$ is denoted by $G^{1/2}$. Let us now consider the operator $G^{1/2}VG^{1/2}$ in $L_2(\Sigma_0)$. It is obvious that it is positive and symmetric in $L_2(\Sigma_0)$, that it acts from $L_2(\Sigma_0)$ into $W_2^1(\Sigma_0)$ and that the estimate

$$\|\,G^{1/2}VG^{1/2}\varphi\,\|_+ \leqslant C\,\|\,\varphi\,\|_{L_2(\Sigma_0)} \qquad (2.3)$$

holds.

The complete continuity of the operator $G^{1/2}VG^{1/2}$ in $L_2(\Sigma_0)$ follows from Rellich's theorem and the estimate (2.3). Since $\ker G = \ker V = 0$, $\ker G^{1/2}VG^{1/2} = 0$.

It follows from what has been said above that there exists a complete system of eigenfunctions $\{\varphi_k\}$ of the operator $G^{1/2}VG^{1/2}$ which is orthonormalized in $L_2(\Sigma_0)$ and, moreover, $\lambda_k G^{1/2}VG^{1/2}\varphi_k = \varphi_k$ and $0 < \lambda_k \nearrow +\infty$ as $k \to \infty$.

The estimate

$$\lambda_n = 0\,(n^{1/2}) \qquad (2.4)$$

is a consequence of (2.3) and the results of /7/.

Let us now consider the functions $\psi_k = G^{1/2}\varphi_k$. It is seen that ψ_k are the eigenfunctions of the operator GV and that λ_k are the eigenvalues corresponding to them. Since $[\psi_k, \psi_m] = [G\varphi_k, \varphi_m] = (\varphi_k, \varphi_m)_{L_2(\Sigma_0)} = \delta_{km}$, the ψ_k from an orthonormalized system in $W_2^1(\Sigma_0)$. It is possible to prove its completeness.

So, the operator GV possesses a complete system of eigenfunctions $\{\psi_k\}$ which are orthonormalized in $W_2^1(\Sigma_0)$ while the eigenvalues corresponding to ψ_k, λ_k, are positive and the estimate (2.4) is valid for them.

Let us now turn to Eq.(2.2). We shall seek its solution in the form

$$\nu(x',t) = \sum_{k=1}^{\infty} c_k(t)\,\psi_k(x')$$

We substitute the latter series into (2.2) and multiply by $\psi_n(x')$ in the sense of a scalar product in $W_2^1(\Sigma_0)$. As a result, we arrive at the problem

$$c_n(t) - \omega_0 S_{\omega_0 t} * c_n(t) + \lambda_n^{-1}\left(\frac{d^2}{dt^2} + \omega_0^2\right)c_n(t) = \Phi_n(t)$$
$$c_n(0) = \frac{d}{dt}\,c_n(t=0) = 0 \qquad (2.5)$$
$$\Phi_n(t) = [Gf, \psi_n] = (f, \psi_n)_{L_2(\Sigma_0)}$$

This problem is solved with the help of the Laplace transformation and the solution has the form /3/

$$c_n(t) = \lambda_n \int_0^t R_n(t-\tau)\,\Phi_n(\tau)\,d\tau \qquad (2.6)$$

$$R_n(t) = \frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} e^{pt} p\,(p^2+\omega_0^2)^{-1/2}(p\,(p^2+\omega_0^2)^{1/2}+\lambda_n)^{-1/2}\,dp, \quad \varkappa>0$$

The branch of the root $(p^2+\omega_0^2)^{1/2}$ is selected such that $(p^2+\omega_0^2)^{1/2}|_{p=0+0}=\omega_0$ and the cut joining the branch points $\pm i\omega_0$ is the segment of the imaginary axis $[-i\omega_0, i\omega_0]$.

With this choice of the branch of $(p^2+\omega_0^2)^{1/2}$, $p=\pm iq_k$ serve as the roots of the equation $p\,(p^2+\omega_0^2)^{1/2}+\lambda_k=0$ where $q_k=((\omega_0^2+(\omega_0^4+4\lambda_k^2)^{1/2})/2)^{1/2}$. Let us now also introduce the quantities $\nu_k=(((\omega_0^4+4\lambda_k^2)^{1/2}-\omega_0^2)/2)^{1/2}$.

Using Jordan's lemma, we transform $R_k(t)$ to the form

$$R_k(t) = R_k^{(1)}(t) + R_k^{(2)}(t)$$

$$R_k^{(1)}(t) = \frac{\lambda_k}{\pi}(\omega_0^4+4\lambda_k^2)^{-1/2}\Big[\int_{-\omega_0}^{\omega_0}(\omega_0^2-\mu^2)^{-1/2}(\mu^2-q_k^2)^{-1}\mu\,\times$$

$$\sin(\mu t)\,d\mu - \int_{-\omega_0}^{\omega_0}(\omega_0^2-\mu^2)^{-1/2}(\mu^2+\nu_k^2)^{-1}\mu\sin(\mu t)\,d\mu\Big]$$

$$R_k^{(2)}(t) = 2q_k(\omega_0^4+4\lambda_k^2)^{-1/2}\sin(q_k t)$$

This transformation enables one to represent $c_n(t)$ in the form

$$c_n(t) = c_n^{(1)}(t) + c_n^{(2)}(t)$$

$$c_n^{(i)}(t) = \lambda_n\int_0^t R_n^{(i)}(t-\tau)\,\Phi_n(\tau)\,d\tau$$

It can be shown that

$$\nu(x',t) = \sum_{k=1}^{\infty} c_k(t)\,\psi_k(x') \in C_0^{(2)}([0,T], W_2^1(\Sigma_0))$$

when

$$f(x',t) \in C^{(3)}([0,T], W_2^{-1}(\Sigma_0)) \cap C_0^{(2)}([0,T], W_2^{(1)}(\Sigma_0))$$

According to the construction it is seen that the density $\nu(x',t)$ satisfies Eq.(2.2). Hence, the following theorem holds.

Theorem 2. A solution of the problem posed in Sect.1 exists and has the form

$$u(x,t) = P[\nu](x,t)$$

$$\nu(x',t) = \sum_{k=1}^{\infty} c_k(t)\,\psi_k(x')$$

where $\psi_k(x')$ is an eigenfunction of the operator GV and the $c_k(t)$ are given by formulae (2.6).

We note that, by virtue of Theorem 1, the solution $u(x,t)$ is unique.

3. The behaviour of the solution at long times. Before engaging in a discussion of the behaviour of the solution at long times, let us dwell on its structure and represent $u(x,t)$ in the form

$$u(x,t) = u_1 + u_2 + v$$

$$u_1 = P[\nu^{(1)}](x,t), \quad u_2 = P_1[\nu^{(2)}](x,t), \quad v = P_2[\nu^{(2)}](x,t)$$

$$\nu^{(i)} = \sum_{k=1}^{\infty} c_k^{(i)}(t)\,\psi_k(x') \quad (i=1,2)$$

It can be shown that $|u_1(x,t)| \to 0$ uniformly with respect to $t\in[0,T]$ as $\omega_0\to 0$ at any point $x\in R_-^3$ for any $T>0$ while, when $\omega_0=0$, $P[\nu^{(2)}](x,t)$ is a solution of the problem on surface gravitational and capillary waves in an ideal stratified liquid. Hence, $u_1(x,t)$ describes internal waves. Since

$$P[\nu^{(2)}](x,t)|_{\omega_0=0} = P_1[\nu^{(2)}](x,t)|_{\omega_0=0} = u_2|_{\omega_0=0}$$

$u_2(x,t)$ corresponds to the surface waves while $v(x,t)$ has the sense of the contribution of the surface waves to the internal waves.

Let us now consider the behaviour of the solution at long times. By virtue of Theorem 2, this is a problem about the properties of $P[\nu](x,t)$, where $\nu(x',t)$ is the density constructed above. The properties of $P[\nu](x,t)$ with a density having a similar structure have been studied in /3/.

Below, we present results, the proofs of which with minor changes repeat the corresponding proofs of the paper cited in the previous footnote.

Theorem 3. Let the function $f(x', t) \in C_0^\infty([0, \infty), W_2^{-1}(\Sigma_0))$ and be finite with respect to time, that is, there exists a T_0 such that $f(x', t) = 0$, $\forall t \geqslant T_0$. Then,

$$|u_2(x, t) + v(x, t) - u^{(0)}(x, t)| \leqslant Ct^{-1/2}, \quad x \in R_-^3, \quad t > 0$$

$$u^{(0)}(x, t) = \sum_{n=1}^{\infty} \frac{q_n^2 \lambda_n F_n(x)}{(\omega_0^4 + 4\lambda_n^2)} \int_0^t \sin(q_n(t-\tau)) \Phi_n(\tau)\, d\tau$$

$$F_n(x) = \frac{1}{\pi} \int_{\Sigma_0} \psi_n(y') (q_n^2 |x' - y'|^2 + (q_n^2 - \omega_0^2) x_3^2)^{-1/2} dy'$$

(We denote all constants encountered in the text by C).

Theorem 4. Let $f(x', t) \in C^3([0, \infty), W_2^{-1}(\Sigma_0)) \cap C_0^{(2)}([0, \infty), W_2^{-1}(\Sigma_0))$ and $f(x', t) = 0$, $\forall t \geqslant T_0$.

Then,

$$|u_1(x, t)| \leqslant Ct^{-1/2}, \quad x \in R_-^3, \quad t > 0$$

It follows from Theorems 3 and 4 that, when a perturbation is finite with respect to time, the solution of the problem does not tend to zero as $t \to \infty$ and has the form $u^{(0)}(x, t) + O(t^{-1/2})$. Moreover, $u^{(0)}(x, t)$ is solely formed due to the surface waves $u_2(x, t)$ and their contribution to the internal waves $v(x, t)$. When $t \geqslant T_0$, the time factor of the terms in the series $u^{(0)}(x, t)$ have the following structure:

$$\sin(q_n t) \int_0^{T_0} \Phi_n(\tau) \cos(q_n \tau)\, d\tau - \cos(q_n t) \int_0^{T_0} \Phi_n(\tau) \sin(q_n \tau)\, d\tau$$

Hence, $u^{(0)}(x, t)$ is the superposition of an infinite number of vibrations with frequencies $\pm q_n$ and describes some residual unsteady process.

Let us now consider a periodic mode of excitation.

Theorem 5. Let $f(x', t) = \eta(t) f(x') e^{-i\omega t}$, where $f(x') \in W_2^{-1}(\Sigma_0)$, $\eta(t)$ is an embedding function possessing the following properties: $\eta(0) = 0$, $\eta(t) = 1$ when $t \geqslant T_0 > 0$, $d\eta(t)/dt \in C_0^\infty[0, \infty)$. The following assertions then hold.

1°. If $\omega > \omega_0$ and $\omega \neq q_n$, $\forall n \in \mathbf{N}$, then

$$|u(x, t) - v^{(0)}(x, t) - w(x) e^{-i\omega t}| \leqslant Ct^{-1/2}, \quad x \in R_-^3, \; t > 0$$

2°. If $0 < \omega < \omega_0$, then

$$|u(x, t) - v^{(0)}(x, t) - w(x) e^{-i\omega t}| \to 0$$

is uniform with respect to $x \in Q$ as $t \to \infty$, where Q is an arbitrary compactum belonging to R_-^3.

3°. If $\omega = q_n$, the solution can be represented in the form

$$u(x, t) = te^{-iq_n t} q_n(x) + U(x, t)$$

$$q_n(x) = \frac{q_n^2 e^{i\pi/2} \Phi_n}{(\omega_0^4 + 4\lambda_n^2)^{1/2}} F_n(x), \quad |U(x, t)| \leqslant C, \quad \forall x \in R_-^3, \quad t > 0$$

The notation

$$v^{(0)}(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} q_n (a_n^+ e^{iq_n t} + a_n^- e^{-iq_n t}) F_n(x)$$

$$a_n^\pm = \frac{q_n \lambda_n \Phi_n}{(\omega_0^4 + 4\lambda_n^2)^{1/2}} \int_0^{T_0} e^{\mp i(q_n \pm \omega)\tau} (q_n \mp \omega)^{-1} \eta'(\tau)\, d\tau$$

$$\Phi_n = [Gf, \psi_n] = (f, \psi_n) L_2(\Sigma_0)$$

$$w(x) = \frac{\omega}{2\pi} \int_{\Sigma_0} g(y') (\omega^2 |x' - y'|^2 - (\omega_0^2 - \omega^2) x_3^2)^{-1/2} dy'$$

$$g(y') = \sum_{n=1}^{\infty} \frac{\Phi_n \psi_n(y') \lambda_n \omega}{(\omega^2 - \omega_0^2)^{1/2}} (\lambda_n - \omega(\omega^2 - \omega_0^2)^{1/2})^{-1}$$

was used above.

The functions $F_n(x)$ are the same as those in Theorem 3.

Let us now discuss the results of Theorem 5. It follows from these results that the solution of the problem does not have a limiting amplitude, that is, in the case of an excitation $f(x', t) = f(x')\eta(t)e^{-i\omega t}$, where $\eta(t)$ is an embedding function, no limit of the expression $e^{i\omega t}u(x, t)$ exists as $t\to\infty$. However, if the background $v^{(0)}(x, t)$ is excluded from the solution $u(x, t)$, then, when $\omega \neq q_n$ and $\omega \neq \omega_0$, the above-mentioned limit will exist and, moreover, the nature of the convergence will depend on the ratio of the frequencies ω and ω_0.

The background $v^{(0)}(x, t)$ is the superposition of an infinite number of vibrations with frequencies $\pm q_n$ and is formed solely from the surface waves $u_2(x, t)$ and their contribution to the internal waves $v(x, t)$ (see /3/). The limiting regime $w(x)\exp(-i\omega t)$ results from both the waves $u_2(x, t)$ and $v(x, t)$ and the internal waves $u_1(x, t)$. When $\omega = q_n$, a linear increase in the amplitude, that is, resonance, is observed. It can be shown that, in this situation, $|u_1(x, t)| < C$ when $(x, t) \in R_-^3 \times [0, \infty)$ and it is therefore correct to speak about the resonance of surface waves. The resonance frequencies form the sequence $\{q_n\}$. It follows from q_n and the estimates (2.4) that $q_n > \omega_0$ and $q_n = O(n^{1/4})$. The numerical values of these frequencies are determined by the magnitudes of the Vaisala-Brunt frequency, ω_0 and the eigenvalues λ_n of the operator GV which depends on the shape of the domain Σ_0 and the capillarity.

As can be seen /3/, there is no substantial difference in the behaviour of the internal waves in the gravitational and gravitational-capillary cases. This does not hold in the case of surface waves since their limiting regimes are strongly dependent on the eigenvalues of the operator GV, which has a different form in each of the above-mentioned cases. In particular, surface wave resonances will be observed at different frequencies in the gravitational and gravitational-capillary cases.

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